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BEM ANALYSIS OF FREE VIBRATION PROBLEMS USING POLYNOMIAL PARTICULAR INTEGRALS

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Abstract—The development of a boundary element method for free vibration analysis using particular solutions is discussed. The formulation presented is based on treating the forcing function, in the governing differential equation, as an initially unknown distributed body force within the domain. The volume integral due to this body force is then eliminated by approximating the inertia force by global interpolation and polynomial function representations and finding particular solutions for the governing, inhomogeneous equations. The resulting non-symmetric system matrix is solved by using a modified Arnoldi's algorithm that takes advantage of the special structure of the substructured boundary element method. The techniques described here are embedded in a computer program GPBEST, and the numerical examples are solved by using this computer code.

1. INTRODUCTION

Generally, the design and development of complex structures require not only the elastic and inelastic responses but also the determination of natural frequencies and associated mode shapes. Until recently the computation of natural frequencies has been carried out mostly by using the finite element method since the solution technique based on the boundary element method (BEM) has been limited. That is, the boundary element formulations were developed by using frequency dependent fundamental solutions, consequently, the procedure required the formation and solution of matrix equations that depend implicitly on the assumed frequency. Recently, a method based on the use of static fundamental solutions coupled with the approximation of the inertia force by global interpolation functions has been developed to solve two-dimensional (Nardini and Brebbia, 1982; Ahmad and Banerjee, 1986), axisymmetric (Wang and Banerjee, 1988, 1990) and three-dimensional (Wilson *et al.*, 1990) free vibration problems. By using the static fundamental solutions, the system equation can be reduced to the form used in the finite element eigenvalue analysis, except that the matrix in BEM is non-symmetrical. A major feature of these developments is the use of global interpolation for the approximation of the inertia force in the governing differential equation. However, considerable advantage may be gained by using the newly developed piece-wise polynomial functions for the approximation of the forcing function as described in this text.

This paper reports a systematic development of boundary element techniques for the solution of free vibration problems in isotropic bodies. Specifically, the solution of free vibration problems using polynomial function approximation is reported for the first time here. Since polynomial as well as global interpolation functions are used for the approximation of the forcing term, the derivation of particular solutions for both cases is detailed. The applicability of the BEM to these problems is demonstrated by solving a number of examples and comparing the solutions to existing finite element results. The effect of the approximation used for the representation of the inertia force is studied by comparing BEM solutions obtained from global interpolation and piece-wise polynomial functions.

2. BOUNDARY INTEGRAL EQUATIONS

For a homogeneous elastic body subjected to harmonic excitation, the equilibrium equation, in the absence of body forces, is

$$(\lambda+\mu)u_{j,ij}(x)+\mu u_{i,jj}(x)+\rho\omega^2 u_i(x) = 0 \tag{1a}$$

where λ, μ are Lamé material constants, ρ is the density and u_i is the displacement tensor. The above equation can be written in the operator form as

$$L_{ij}u_j(x)+\rho\omega^2 u_i(x) = 0 \tag{1b}$$

where L_{ij} is the Navier operator.

Generally, two different boundary element method approaches have been practised for the calculation of natural frequencies. The first involves the conversion of the differential equation to a boundary integral equation (BIE) in which the frequency ω, appears non-linearly in the system matrix. This can be done either by using the complex point load solution to the forced response problem (Tai and Shaw, 1979; Niwa *et al.*, 1982) or by using arbitrary singular solutions in terms of real variables (DeMey, 1976a,b). Notwithstanding the type of fundamental solution used, the boundary integral equation, after suitable approximation of geometry and field variables by interpolation functions, reduces to a homogeneous set of algebraic equations as

$$[A(\omega)]\{X\} = \{0\}. \tag{2}$$

The natural frequencies of the system are determined by solving

$$\det[A(\omega)] = 0. \tag{3}$$

Since the matrix $[A]$ depends on the frequency, some form of determinant search procedure must be employed. This method has two major drawbacks: (a) the method requires repeated formation of system matrix for different values of frequencies, which makes the technique extremely inefficient, and (b) the method can easily fail for closely spaced roots.

In the second BEM approach the forcing term is treated as a distributed body force in a homogeneous medium. That is, a boundary element formulation can be developed by converting the differential equation (1) to an integral equation in which the inertia force is treated as an initially unknown body force. The boundary integral equation then becomes (Banerjee and Butterfield, 1981)

$$C_{ij}(\xi)u_i(\xi) = \int_S G_{ij}(x,\xi)t_i(x)\,dS(x) - \int_S F_{ij}(x,\xi)u_i(x)\,dS(x) + \rho\omega^2\int_V G_{ij}(z,\xi)u_i(z)\,dV(z) \tag{4}$$

where t_i is the traction vector at the surface, C_{ij} is a matrix that arises due to the singularity of the F_{ij} kernel at a boundary point ξ, and G_{ij} and F_{ij} are the fundamental displacement and traction solutions, respectively, due to a unit point load in a homogeneous elastic body. The expressions for the fundamental solutions can be found in Banerjee and Butterfield (1981).

Let us assume that the displacement in the forcing function can be approximated by a series. Although the displacement can be approximated by infinitely many functions, two general classes are used in the current analysis. In the first representation the displacement in the inertia term, at the field point or sampling point x, is approximated in terms of a function that depends on the positive powers of the distance between the field point and source point ξ_m, as

$$u_i(x) \approx \hat{u}_i(x) = \sum_{m=1}^{\infty} K(x, \xi_m)\phi_i(\xi_m) \tag{5a}$$

where $K(x, \xi_m)$ is a function of the distance between the field and source points and ϕ_i are unknown coefficients associated with the source point ξ_m.

In the second approach the displacement, in the forcing term, is approximated by functions of complete polynomials. The displacement is then expressed as

$$u_i(x) \approx \hat{u}_i(x) = \sum_{m=1}^{\infty} K_m(x)\phi_{im} \tag{5b}$$

where K_m is a vector of polynomial functions and ϕ_{im} is the vector of the unknown coefficients associated with the polynomial terms. It should be noted that, in the above representations, all components of the displacement vector are approximated by the same function.

Based on the approximation, a particular solution to the governing inhomogeneous differential equation (1) can be found from

$$L_{ij} u_j^0(x) + \rho\omega^2 \hat{u}_i(x) = 0 \tag{6}$$

where the particular solution is denoted by superscript 0. Noting the similarity between eqns (6) and (1) an integral equation for the particular solution field, described by eqn (6), can be obtained as

$$C_{ij}(\xi)u_i^0(\xi) = \int_S G_{ij}(x,\xi)t_i^0(x)\,\mathrm{d}S(x) - \int_S F_{ij}(x,\xi)u_i^0(x)\,\mathrm{d}S(x) + \rho\omega^2\int_V G_{ij}(z,\xi)\hat{u}_i(z)\,\mathrm{d}V(z). \tag{7}$$

By subtracting eqn (7) from eqn (4), a modified integral equation is obtained as

$$C_{ij}(\xi)u_i(\xi) - \int_S G_{ij}(x,\xi)t_i(x)\,\mathrm{d}S(x) + \int_S F_{ij}(x,\xi)u_i(x)\,\mathrm{d}S(x)$$
$$= C_{ij}(\xi)u_i^0(\xi) - \int_S G_{ij}(x,\xi)t_i^0(x)\,\mathrm{d}S(x) + \int_S F_{ij}(x,\xi)u_i^0(x)\,\mathrm{d}S(x) + E_j(\xi) \tag{8a}$$

where the residual vector E_j is given by

$$E_j(\xi) = \rho\omega^2 \int_V G_{ij}(z,\xi)[u_i(z) - \hat{u}_i(z)]\,\mathrm{d}V(z). \tag{8b}$$

The residual vector E_j is made negligible by approximating the displacement, in the inertia term, by suitable functions, thereby arriving at a surface only integral equation.

3. PARTICULAR SOLUTIONS

The formulation developed in the previous section requires the knowledge of particular solutions of the governing inhomogeneous differential eqn (6). Although the use of particular solutions is a classical technique, dating to the beginning of the systematic study of linear ordinary differential equations, it appears that the first application of the method of particular solutions in the context of boundary element method was presented by Jaswon and Maiti (1968) for the solution of plate problems under normal loading. Later, the use of particular solutions for the analysis of centrifugal and gravitational body forces was tentatively discussed by Lachat and Watson (1976). Banerjee and co-workers have extended this method to the solution of a wide class of body force problems (Pape and Banerjee, 1987; Henry *et al.*, 1987; Banerjee *et al.*, 1988; Henry and Banerjee, 1988a,b). Previous application of this procedure for free vibration problems includes the work of Ahmad and Banerjee (1986) for two-dimensional problems, Wang and Banerjee (1988, 1990) for axi-symmetric and generalized axi-symmetric problems and finally Wilson *et al.* (1990) for three-dimensional free vibration problems.

The determination of particular solutions requires solutions of the inhomogeneous differential eqn (6). The forcing function, which is unknown for the free vibration problem, can be approximated by primitive functions such as the ones given by eqn (5) and the particular solutions are derived with respect to these primitive functions. Appropriate selection of functions, for the approximation of the inertia term, forces the residuals involving the volume integrals in eqn (8) to be negligible. Therefore, the selection of the primitive function is vital for the accuracy of the solutions. Although infinitely many functions may be selected for the approximation of the forcing term, two general classes of functions are discussed in this paper. The first type of function, which was originally used by Lord Rayleigh (1896), is based on expressing the displacement in the forcing function by an arbitrary variation in positive powers of the distance between the origin and a point ξ_m. In particular, a function that varies linearly with the distance r, is constructed as

$$\hat{u}_i(x) = \sum_{m=1}^{M} [R - r(x, \xi_m)]\phi_i(\xi_m) \tag{9}$$

where $r(x, \xi_m)$ is the distance between the source point ξ_m, and the field point or sampling point x, at which the particular solutions are to be evaluated, R is an arbitrary constant such as the largest dimension of the problem and $\phi_i(\xi_m)$ is an unknown coefficient associated with the source point ξ_m. It should be noted that in the numerical implementation, the infinite series is replaced by a finite series, where M is the total number of terms in the series. This type of interpolation function was used in all previous eigenvalue analyses.

The second primitive function used in the current analysis, for the approximation of the forcing term, is based on complete polynomials. It is known that polynomials form a complete set over the cube in three-dimensional space (Davis, 1963). This can be trivially extended to any rectangular parallelepiped. Further, this set is also complete for any body which can be embedded in a rectangular parallelepiped and whose displacement field can be continuously extended to the parallelepiped. In this case the error between the true solution and the approximate solution can be shown to vanish as the number of terms in the expansion approaches infinity (Wilson *et al.*, 1990). Here again, the infinite series is replaced by a finite number of polynomial terms, where M is the number of terms in the polynomial series.

Assuming complete polynomials of order 2, the polynomial function can be expressed in terms of Cartesian coordinates as

$$\hat{u}_i(x) = \sum_{m=1}^{M} K_m(x)\phi_{im} \tag{10}$$

where

$$K_m = \{1 \quad x_1 \quad x_2 \quad x_3 \quad x_1^2 \quad x_2^2 \quad x_3^2 \quad x_1x_2 \quad x_1x_3 \quad x_2x_3\} \quad \text{for} \quad 3\text{D}$$

$$K_m = \{1 \quad x_1 \quad x_2 \quad x_1^2 \quad x_2^2 \quad x_1x_2\} \quad \text{for} \quad 2\text{D}$$

and ϕ_{im} are the associated unknown coefficients.

The functions used for the approximation of the displacement in the forcing term can be used in two different ways: (a) a suitable combination of functions may be used to approximate the forcing term throughout a subregion, or an entire region, and (b) a relatively small set of functions can be associated at specific geometric locations in the region and the overall approximation is built up as a sum of these point based functions.

Global interpolation

The global interpolation method has been discussed previously for two- (Ahmad and Banerjee, 1986) and three-dimensional (Wilson *et al.*, 1990) problems. However, to illustrate the difference between the global interpolation and polynomial function approximations the derivation of particular solutions, based on global interpolation, is also included here. Moreover, the particular solutions are derived systematically through the use of the Galerkin vector, thus, the particular solutions presented here are different from the ones reported previously. Nevertheless, since particular solutions are non-unique, the final results are not affected by the use of these new particular solutions.

The determination of particular solutions, for the free vibration analysis, is facilitated by the use of the Galerkin vector g_i, which is related to displacements by (Fung, 1965)

$$u_i^0 = a g_{i,jj} - b g_{j,ij} \tag{11a}$$

where, in terms of shear modulus μ and Poisson's ratio v,

$$a = \frac{1-v}{\mu}$$

and

$$b = \frac{1}{2\mu}.$$

Substituting the above Galerkin vector in the governing differential equation (6), we get

$$(1-v) g_{i,jjkk} + \rho\omega^2 \hat{u}_i = 0. \tag{11b}$$

The displacement particular solution is then determined by approximating the displacement, in the inertia term, by eqn (9) to arrive at

$$u_i^0(x) = \rho\omega^2 \sum_{m=1}^{M} U_{ij}(x, \xi_m)\phi_j(\xi_m) \tag{11c}$$

where

$$U_{ij}(x, \xi_m) = (D_1 + D_2 r)\delta_{ij} r^2 + (D_3 + D_4 r) y_i y_j$$

$$D_1 = -\frac{2(d+2)(1-v)-1}{60\mu(1-v)} R, \quad D_2 = \frac{2(d+3)(1-v)-1}{144\mu(1-v)}$$

$$D_3 = \frac{1}{30\mu(1-v)} R, \quad D_4 = -\frac{1}{48\mu(1-v)}$$

and d is the dimensionality of the problem.

Even though all components of the displacement vector are approximated by the same function the displacement particular solution is coupled; that is, each component of the displacement particular solution is related to all components of the ϕ_i vector. The same is also true for particular solutions based on polynomial functions.

Using the constitutive relationship $\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})$, the stress particular solution is obtained as

$$\sigma_{ij}^0(x) = \rho\omega^2 \sum_{m=1}^{M} S_{ijk}(x, \xi_m)\phi_k(\xi_m) \tag{11d}$$

where S_{ijk} is

$$S_{ijk}(x, \xi_m) = (D_5 + D_6 r)\delta_{ij} y_k + (D_7 + D_8 r)(\delta_{ik} y_j + \delta_{jk} y_i) + D_9 \frac{y_i y_j y_k}{r}$$

$$D_5 = 2\lambda D_1 + [(d+1)\lambda + 2\mu]D_3, \quad D_6 = 3\lambda D_2 + [(d+2)\lambda + 2\mu]D_4,$$
$$D_7 = \mu(2D_1 + D_3), \quad D_8 = \mu(3D_2 + D_4), \quad D_9 = 2\mu D_4.$$

The corresponding traction particular solution is derived from the Cauchy relationship as

$$t_i^0(x) = \rho\omega^2 \sum_{m=1}^{M} T_{ik}(x, \xi_m)\phi_k(\xi_m) \tag{11e}$$

where

$$T_{ik}(x, \xi_m) = S_{ijk}(x, \xi_m) n_j(x).$$

Polynomial function

Based on the representation given by eqn (10), a displacement particular solution, for the three-dimensional problem, is obtained by using the Galerkin vector as

$$u_i^0(x) = \rho\omega^2 \sum_{m=1}^{M} U_{ijm}(x)\phi_{jm} \tag{12a}$$

where

$$U_{ij1} = 12B_1[(a\delta_{ij} - b\delta_{i1}\delta_{j1})x_1^2 + (a\delta_{ij} - b\delta_{i2}\delta_{j2})x_2^2 + (a\delta_{ij} - b\delta_{i3}\delta_{j3})x_3^2],$$
$$U_{ijm} = 20B_2(a\delta_{ij} - b\delta_{in}\delta_{jn})x_n^3, \quad m = 2, 3, 4, \quad n = m-1,$$
$$U_{ijm} = 30B_3(a\delta_{ij} - b\delta_{in}\delta_{jn})x_n^4, \quad m = 5, 6, 7, \quad n = m-4,$$
$$U_{ijm} = 3B_4[2a\delta_{ij}(x_n^2 + x_l^2) - b\{2\delta_{il}\delta_{jl}x_n^2 + 3(\delta_{in}\delta_{jl} + \delta_{il}\delta_{jn})x_n x_l + 2\delta_{in}\delta_{jn}x_l^2\}]x_n x_l,$$
$$m = 8(n = 1, l = 2), \quad m = 9(n = 1, l = 3), \quad m = 10(n = 2, l = 3).$$

The constants B_k are given by

$$B_1 = -\frac{1}{24d(1-v)}, \quad B_2 = -\frac{1}{120(1-v)}, \quad B_3 = -\frac{1}{360(1-v)}, \quad B_4 = -\frac{1}{72(1-v)}.$$

Using the derivative of the displacement particular solution and Hooke's law, the stress particular solution is obtained as

$$\sigma_{ij}(x) = \rho\omega^2 \sum_{m=1}^{M} S_{ijkm}(x)\phi_{km} \tag{12b}$$

where S_{ijkm} is

$$S_{ijk1} = 24B_1[\lambda(a-b)\delta_{ij}x_k + \mu\{a(\delta_{ik}x_j + \delta_{jk}x_i) - 2b(\delta_{i1}\delta_{j1}\delta_{k1}x_1 + \delta_{i2}\delta_{j2}\delta_{k2}x_2 + \delta_{i3}\delta_{j3}\delta_{k3}x_3)\}]$$
$$S_{ijkm} = 60B_2 x_l^2[\lambda(a-b)\delta_{ij}\delta_{kl} + \mu\{a(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}) - 2b\delta_{il}\delta_{jl}\delta_{kl}\}], \quad m = 2, 3, 4, \quad l = m-1$$
$$S_{ijkm} = 120B_3 x_l^3[\lambda(a-b)\delta_{ij}\delta_{kl} + \mu\{a(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}) - 2b\delta_{il}\delta_{jl}\delta_{kl}\}], \quad m = 5, 6, 7, \quad l = m-4$$

$$
\begin{aligned}
S_{ijkm} = 6B_4[&\lambda(a-b)\delta_{ij}[\delta_{kn}x_l^3+3\delta_{kl}x_l^2x_n+3\delta_{kn}x_lx_n^2+\delta_{kl}x_n^3] \\
&+\mu[a\{(\delta_{ik}\delta_{jn}+\delta_{jk}\delta_{in})x_l^3+3(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})x_l^2x_n+3(\delta_{ik}\delta_{jn}+\delta_{jk}\delta_{in})x_lx_n^2 \\
&+(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})x_n^3\}-2b\{\delta_{in}\delta_{jn}\delta_{kn}x_l^3+3(\delta_{in}\delta_{jn}\delta_{kl}+\delta_{il}\delta_{jn}\delta_{kn} \\
&+\delta_{in}\delta_{jl}\delta_{kn})x_l^2x_n+3(\delta_{il}\delta_{jl}\delta_{kn}+\delta_{il}\delta_{jn}\delta_{kl}+\delta_{in}\delta_{jl}\delta_{kl})x_lx_n^2+\delta_{il}\delta_{jl}\delta_{kl}x_n^3\}]], \\
&m = 8(l=1,n=2),\quad m=9(l=1,n=3),\quad m=10(l=2,n=3).
\end{aligned}
$$

The traction particular solution is then obtained from the stress solution as

$$t_i^0(x) = \rho\omega^2 \sum_{m=1}^{M} T_{ikm}(x)\phi_{km} \tag{12c}$$

where

$$T_{ikm}(x) = S_{ijkm}(x)n_j(x).$$

In the above equations summation over indices i, j, k are implied. For the three-dimensional problem, $i, j, k = 1, 2, 3$.

Since the primitive function, for the two-dimensional problem, is a subset of the three-dimensional function, the two-dimensional particular solutions are obtained from the corresponding three-dimensional solutions by setting components in the third direction to zero. For the two-dimensional case the indices i, j, k take the values $i, j, k = 1, 2$ and the components in the third direction are set to zero.

Only polynomials of order 2 are used in the present analysis. While higher order polynomials can be constructed, it is unlikely that these polynomials would be of much practical use. This is true due to the difficulty of fitting the forcing term distribution to multi-dimensional, higher order polynomials and also due to the well-known tendency of these polynomials becoming unstable. However, since only polynomials of the second order are used for the approximation, accurate evaluation of eigenfrequencies, especially higher modes, requires a subdivision of the problem domain so that within each domain the variation of the inertia force is adequately approximated.

Determination of unknown coefficients

Notwithstanding the type of representation used for the approximation of the displacement in the forcing function, the displacement in both cases is expressed in terms of unknown coefficients. The elimination of these coefficients requires the evaluation of the interpolation equation (9) or the polynomial function equation (10) at pre-selected sampling points. For example, consider the determination of the unknown coefficients in the global interpolation scheme. By collocating the interpolation equation at all sampling points a set of algebraic equations is obtained as

$$\{\hat{u}\} = [\Delta K]\{\phi\} \tag{13}$$

where $[\Delta K]$ is a matrix of the order $N \times M$; N is the number of sampling points and M is the number of source points. When the sampling points coincide with the source points a square matrix $(N \times N)$ is obtained. This matrix can be inverted to obtain the unknown coefficients as

$$\{\phi\} = [\Delta K]^{-1}\{\hat{u}\} = [\Delta\hat{K}]\{\hat{u}\}. \tag{14}$$

This procedure was adapted in all the free-vibration analyses reported previously using global interpolation function. It should be noted that the matrix to be inverted is of the order $(N \times N)$ since all components of the displacement vector are approximated by the same function, therefore, eqn (10) is written only for one component of the displacement vector at each sampling point.

However, when the number of sampling points is different from the number of source points the matrix is non-square. In this case a suitable procedure that can be used for the determination of unknown coefficients is the least square regression approach. The least square approach can be used for both over-determined (number of sampling points greater than number of source points) and under-determined (number of sampling points fewer than number of source points) systems. In the least square approach, an estimate of the unknown coefficients $\{\phi\}$, is given by

$$\{\phi\} = ([\Delta K^T][\Delta K])^{-1}[\Delta K^T]\{\ddot{u}\} = [\Delta\hat{K}]\{\ddot{u}\}. \tag{15}$$

For the polynomial function approximation, the evaluation of unknown coefficients, in general, requires the solution of an over-determined system. That is, for polynomial function of order two $M = 6$, for the two-dimensional situation and $M = 10$, for the three-dimensional case, therefore, for all realistic engineering problems, the number of unknown coefficients M will be less than the number of sampling points N. The unknown coefficients in this case are also determined by using the least square regression approach.

Note that instead of inverting an $N \times N$ matrix, as in the normal global interpolation approach, the least square estimation requires the inversion of an $M \times M$ matrix. Therefore, the polynomial function procedure is much more efficient than the previously developed global interpolation technique, as will be confirmed by the numerical examples.

4. FORMULATION OF THE ALGEBRAIC EIGENVALUE PROBLEM

An approximate analysis based on the integral equation (8a) requires only the modeling of the surface of the problem domain. By dividing the surface into a series of boundary patches and approximating geometry and field variables within each boundary patch using shape functions, the discretized integrals can be evaluated. In the present analysis, the geometry, physical and particular field variables are approximated by isoparametric quadratic shape functions and the resulting discretized integrals are evaluated numerically. Evaluation of these boundary integrals at all surface nodes leads to the algebraic equivalent,

$$[\Delta G]\{t\} - [\Delta F]\{u\} = [\Delta G]\{t^0\} - [\Delta F]\{u^0\}. \tag{16}$$

It is implied that the physical and particular solution fields are approximated by the same shape functions. While this is not a requirement, the representation of physical and particular solution fields by the same shape function makes the method computationally efficient. To illustrate, consider the discretized integrals, corresponding to a three-dimensional problem, mapped over a unit element, i.e. a discretized form of eqn (8a),

$$\begin{aligned} C_{ij}(\xi)u_i(\xi) &- \sum_{m=1}^{M}\int_{-1}^{1}\int_{-1}^{1} G_{ij}(x(\eta_1,\eta_2),\xi)J(\eta_1,\eta_2)N_1^{\alpha}(\eta_1,\eta_2)\,d\eta_1\eta_2 t_i^{\alpha} \\ &+ \sum_{m=1}^{M}\int_{-1}^{1}\int_{-1}^{1} F_{ij}(x(\eta_1,\eta_2),\xi)J(\eta_1,\eta_2)N_1^{\alpha}(\eta_1,\eta_2)\,d\eta_1\eta_2 u_i^{\alpha} \\ &= C_{ij}(\xi)u_i^0(\xi) - \sum_{m=1}^{M}\int_{-1}^{1}\int_{-1}^{1} G_{ij}(x(\eta_1,\eta_2),\xi)J(\eta_1,\eta_2)N_2^{\alpha}\,d\eta_1,\eta_2 t_i^{0\alpha} \\ &+ \sum_{m=1}^{M}\int_{-1}^{1}\int_{-1}^{1} F_{ij}(x(\eta_1,\eta_2),\xi)J(\eta_1,\eta_2)N_2^{\alpha}(\eta_1,\eta_2)\,d\eta_1\eta_2 u_i^{0\alpha} \end{aligned}$$

where J is the Jacobian of transformation, N_1 is the shape function used for the interpolation of physical variables, N_2 is the shape function used for the interpolation of particular solutions, and superscript α indicates quantities related to node α. It is obvious that the integrals on the left-hand and the right-hand sides of the above equation are identical when $N_1 = N_2$. Thus, by choosing the same shape function for the approximation of physical and particular solution fields the integration effort for the eigenvalue problem is not increased from the corresponding requirement for the static problem.

The particular solutions derived in the previous section can be used to reduce eqn (16) to

$$[\Delta G]\{t\}-[\Delta F]\{u\} = \rho\omega^2([\Delta G][\Delta Q]-[\Delta F][\Delta P])\{\phi\}. \quad (17)$$

The unknown coefficients $\{\phi\}$, can be eliminated from the above equation by using eqns (14) or (15). Then eqn (17), in terms of physical variables, becomes

$$[\Delta G]\{t\}-[\Delta F]\{u\} = \rho\omega^2[M]\{u\} \quad (18)$$

where

$$[M] = ([\Delta G][\Delta Q]-[\Delta F][\Delta P])[\Delta \hat{K}].$$

Separation of displacements and tractions into unknown $\{X\}$, and known $\{Y\}$, vectors leads to

$$([A]-\rho\omega^2[\bar{M}])\{X\} = ([B]-\rho\omega^2[\hat{M}])\{Y\} \quad (19)$$

where the mass matrices $[\bar{M}]$ and $[\hat{M}]$ are augmented to the same dimensions as $[A]$ and $[B]$ by adding zeros appropriately.

Since the applied boundary loading $\{Y\}$ is zero in the eigenfrequency analysis, eqn (19), for a single region problem, reduces to a standard eigenvalue expression. In a multi-region analysis both the system matrices $[A]$, $[B]$ and the mass matrices $[\bar{M}]$, $[\hat{M}]$ are assembled in exactly the same manner used in the static analysis. In this case physical field variables at the interface between subregions will generally be non-zero but the interface quantities at adjacent subregions are related through continuity and compatibility conditions.

The eigenvalues of eqn (19) with zero on the right-hand side can be extracted by using routines from EISPACK (Garbow, 1980) software package. However, the computing effort of the extraction procedure, using EISPACK routines, increases rapidly with increasing matrix size. Further, no use is made of the special structure of boundary element matrices or the fact that only a small portion of the spectrum is required. The extraction procedure developed by Wilson *et al.* (1990), based on a modified Arnoldi's algorithm, is used in the present analysis.

5. NUMERICAL EXAMPLES

Cantilever beam (2-*D*)

The two-dimensional free vibration analysis was validated by comparing the first four bending modes of a rectangular fixed end cantilever beam obtained by the finite element method and the boundary element method. The length of the beam is taken as 6.5 units and the cross-section as a 1 unit square. The material parameters used were $E = 10^4$ units,

Table 1. Natural frequencies (Hz) for cantilever beam

Mode	FEM MHOST Wilson (1986)	2 regions GIA	2 regions PFA	4 regions GIA	4 regions PFA
1 (flex)	0.378	0.371	0.376	0.362	0.375
2 (flex)	2.188	2.176	2.223	2.126	2.182
3 (axial)	—	3.791	3.847	3.720	3.748
4 (flex)	5.583	5.536	5.570	5.493	5.589
5 (flex)	9.908	9.779	10.645	9.834	10.081
Time		46	36	54	49

$\nu = 0.3$ and $\rho = 1.0$ unit. Table 1 shows a comparison of various solutions. The BEM solutions agree well with the finite element results, however, accurate evaluation of eigenvalues using the polynomial function approximation required substructuring of the cantilever beam. While the solution time (all reported solution times are in seconds and these were obtained on an HP 9000 series 800 computer) using the polynomial function is less than the corresponding global interpolation based solution time, the difference is not significant due to the small size of the problem. The first four flexural mode shapes (Fig. 1), obtained from the polynomial function based approach using the four region model, are also in agreement with the shape predicted by the beam theory.

Fixed-end arch with and without openings (2-D)

This example is concerned with the evaluation of the eigenvalues of a fixed end arch. Two different cases were studied; in the first case the fixed-end arch was considered without openings and in the second case the arch was considered to have rectangular openings. The material properties were taken as $E = 10^8$ units, $\rho = 1.0$ unit and $\nu = 0.2$. This problem was studied previously by Ahmad and Banerjee (1986), however, the size and location of the rectangular openings used by them are not known and therefore, different dimensions for the openings were used in the present study. Single and multi-region boundary element models were used; the six region models of the arch without the openings and with the openings are shown in Figs 2(a) and 2(b), respectively. The first four modes based on global interpolation and polynomial function approximations of the inertia force, for various boundary element models, are shown in Tables 2(a) and (b). The global interpolation solutions and the polynomial function results using substructured models agree well with

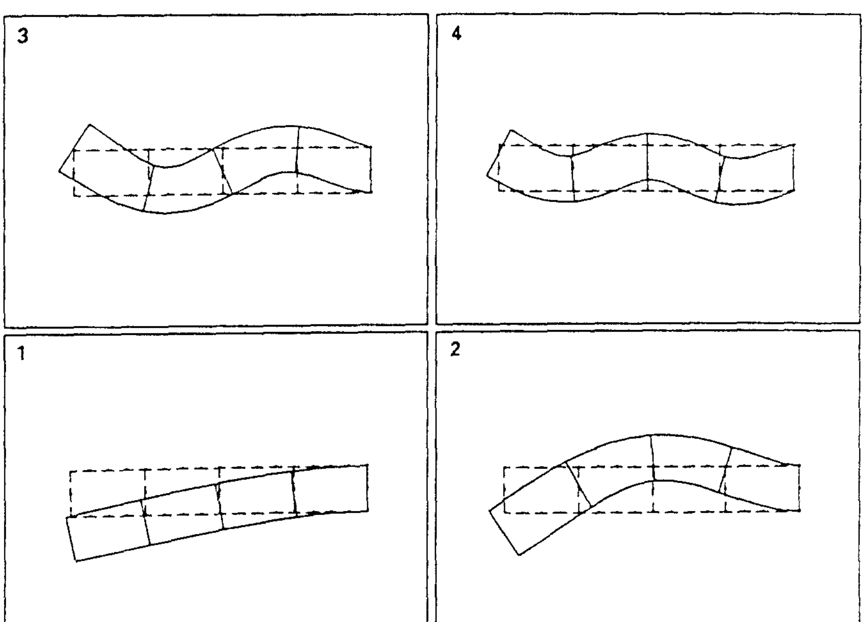
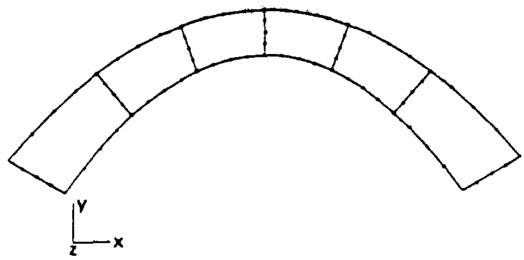
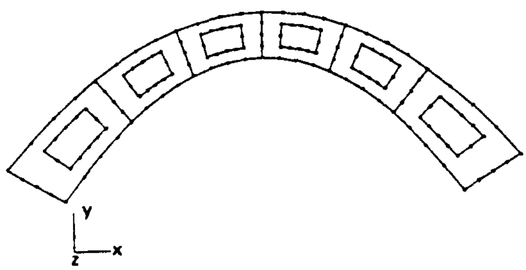


Fig. 1. Flexural mode shapes for cantilever beam.



(a) Without opening



(b) With opening

Fig. 2. Multi-region BEM model for fixed-end arch.

each other. The mode shapes for the six region arch model obtained by using the polynomial function and global interpolation based approaches are identical and the mode shapes based on polynomial functions are shown in Figs 3(a) and 3(b).

Cantilever beam (3-*D*)

The three-dimensional free vibration analysis was validated by computing the natural frequencies of a cantilever beam that is fixed, in all three directions, on one face. The specific dimensions of the problem, studied previously by Leissa and Zhang (1983) using Ritz technique and by Wilson *et al.* (1990) using the boundary element method based on global interpolation, are $1 \times 1 \times 0.5$ as shown in Fig. 4. Two different boundary element models were used. In the first model, side A was divided into three elements, side B was divided into two elements and side C was modeled by one element. In the second model, sides B and C were modeled as before but alongside A, the body was substructured into three regions along the element boundaries. In all models, sides opposite to A, B and C were

Table 2. Natural frequencies (Hz) for fixed-end arch

	1 region	3 regions		6 regions	
Mode	GIA	GIA	PFA	GIA	PFA
(a) without opening					
1	89.3	89.0	90.0	87.6	89.8
2	127.6	126.8	127.8	124.4	127.9
3	178.4	179.5	177.1	176.8	180.4
4	236.2	235.9	239.4	231.8	237.6
Time	155	165	122	255	219
(b) with opening					
1	58.4	56.7	57.6	56.1	56.8
2	100.0	97.2	101.9	97.9	100.1
3	115.1	114.9	116.4	114.2	115.3
4	145.2	144.7	159.0	144.6	150.4
Time	1214	1460	544	946	703

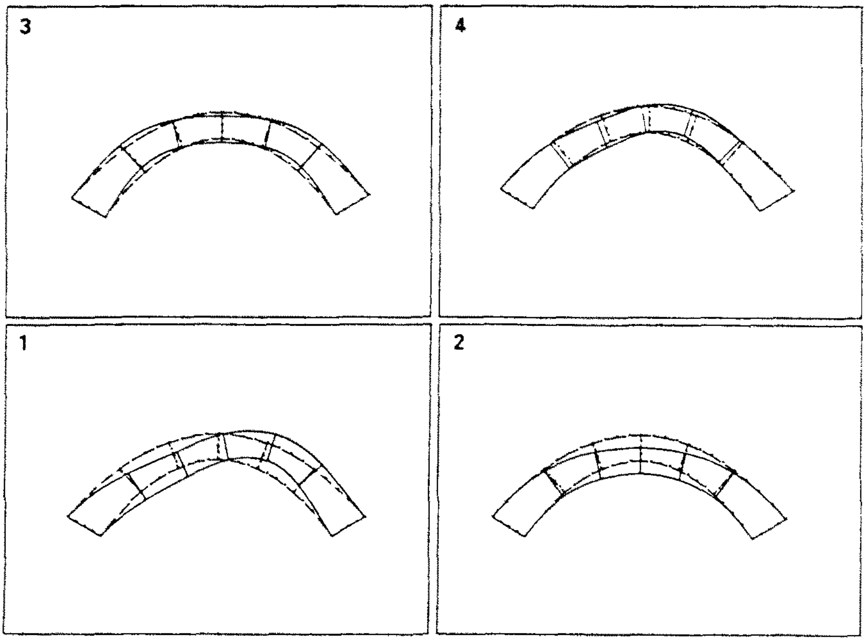


Fig. 3(a). Mode shapes for fixed-end arch without openings.

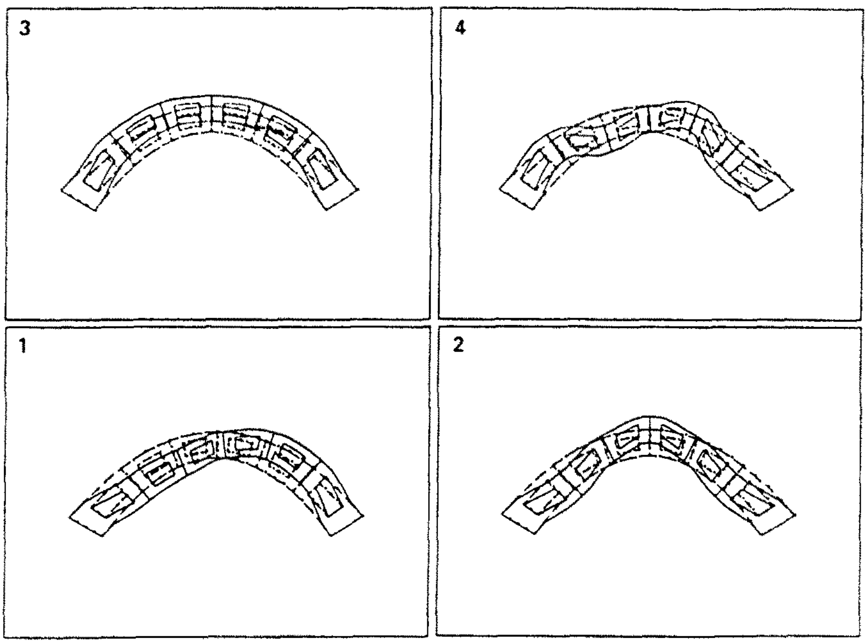


Fig. 3(b). Mode shapes for fixed-end arch with openings.

modeled by the same number of elements. The material properties used were $E = 16.126 \times 10^6$ psi, $\nu = 0.3$ and $\rho = 0.0007$ lb m in^{-3}. The results from the two-boundary element models using both global interpolation and polynomial function approximations of the inertia force are compared to the solutions obtained from Ritz method in Table 3. The frequency results shown were normalized as $\omega\sqrt{\rho/E}$. For the single region model the maximum error using global interpolation is 1.3%, whereas the maximum error using polynomial function approximation is 3.9%. However, using the three-region model the maximum error in the polynomial function approximation case is reduced to 2.0%. The solution times again indicate the advantage of polynomial function approximation procedure. Here again, the mode shapes obtained from global interpolation and polynomial

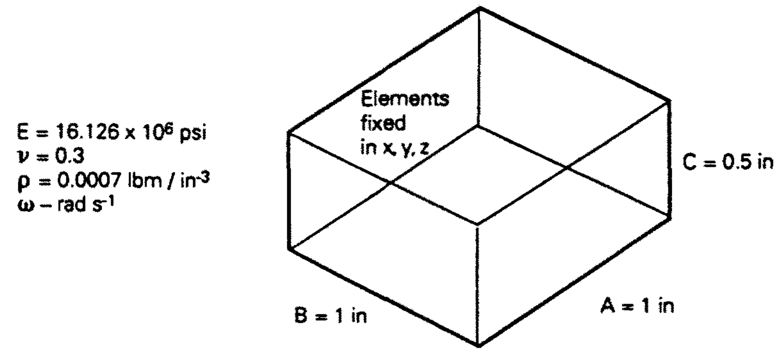


Fig. 4. Geometry of cantilever beam.

Table 3. Natural frequencies for cantilever beam

Mode	Type	Ritz	GIA	PFA 1 reg	PFA 3 reg
1	EB	0.447	0.441	0.455	0.441
2	SB	0.667	0.662	0.679	0.672
3	T	0.788	0.788	0.819	0.805
4	L	1.596	1.688	1.627	1.612
Time			930	438	792

EB—Easy bending, SB—Stiff bending, T—Torsion, L—Extension.

function approximations are in agreement. The first four mode shapes, obtained from polynomial function approximation, are shown in Fig. 5.

Skewed cantilever plates (3-*D*)

The three-dimensional free vibration analysis was further validated by comparing the BEM solutions for natural frequencies of skewed cantilever plates to the results obtained by the Ritz method and the finite element method (McGee and Leissa, 1991). The free

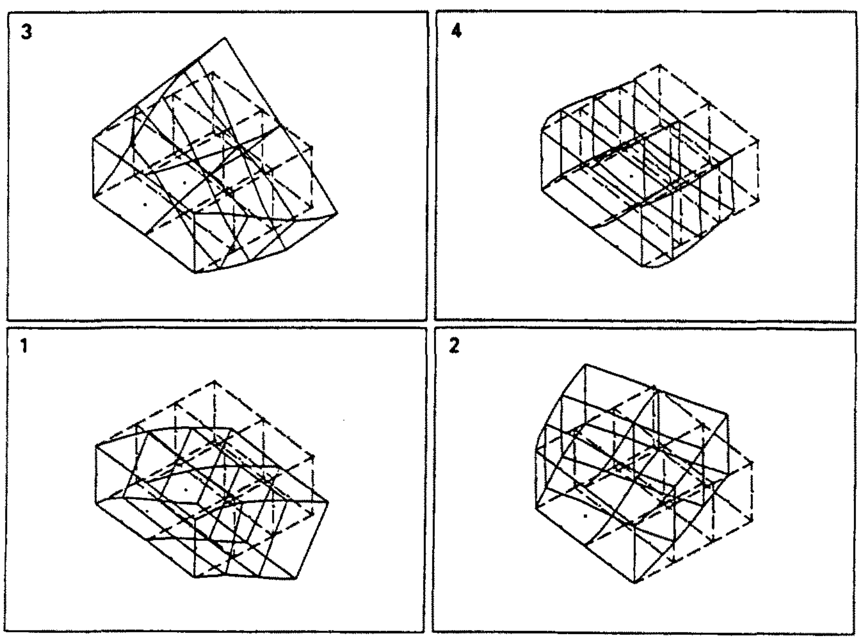


Fig. 5. Mode shapes for rectangular cantilever beam.

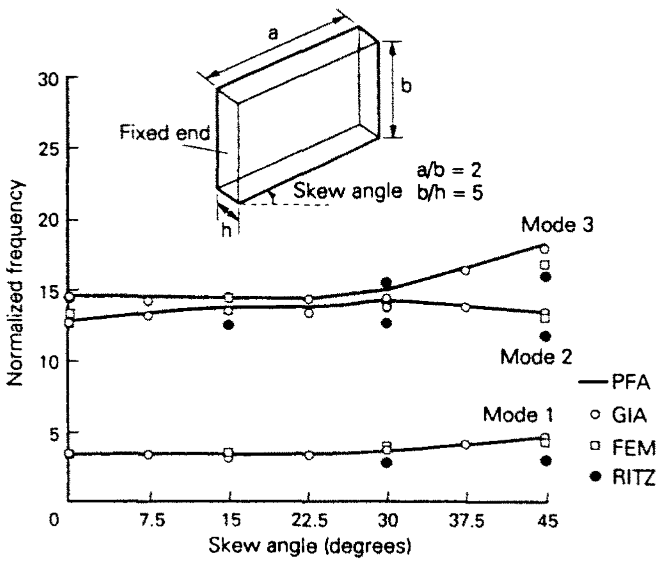


Fig. 6. Comparison of natural frequencies of skewed plate.

vibration characteristics of skewed cantilevered plates are important in the study of aerodynamic lifting or stabilizing surfaces. A particular case of a skewed plate of length to width ratio of 2 and width to thickness ratio of 5 was investigated. The problem was modeled as a single region that consists of four elements along the length, two elements along the width and one element in the thickness direction. A two-region model was also created by dividing the structure into half along the length of the plate and the polynomial function based results were obtained from this model. The boundary element results obtained from global interpolation and polynomial function based approximations agreed well with the results reported by McGee and Leissa (1991). A comparison of the natural frequencies for the first three modes with the corresponding Ritz and finite element results for various skew angles is shown in Fig. 6. The natural frequencies are normalized as $\omega a^2\sqrt{\rho h/D}$, where $D = Eh^3/12(1-\nu^2)$.

6. CONCLUSION

The boundary integral equation for free vibration problems is systematically developed by treating the forcing function in the governing, differential equation as an initially unknown distributed body force. By approximating the forcing function by global interpolation and polynomial function representations and utilizing the particular solutions of the governing inhomogeneous differential equation, a surface only integral equation is derived. The resulting system matrix is similar to the corresponding FEM matrix, except in BEM the matrix is non-symmetrical. The BEM system matrix is solved by using a modified Arnoldi's algorithm developed earlier in Wilson *et al.* (1990). The numerical examples presented indicate the applicability of these BEM techniques for the solution of free vibration problems. The results further indicate that the solution technique based on polynomial function approximation is computationally more efficient than the solution procedure based on global interpolation, however, to obtain accurate results the problem domain needs to be substructured when the polynomial function based approach is used.

The BEM solution techniques presented here may be enhanced by improving certain features of the procedure. In the polynomial function based approach only polynomials of the order 2 were used in the present analysis. It is possible to construct particular solutions for higher order polynomials. While the use of higher order terms, in general, is not practical due to the difficulty of fitting these functions in a multi-dimensional situation and also due to the tendency of higher order polynomials becoming unstable, at least polynomials of the order up to 3–4 may be attempted. This generally will require the introduction of interior points for the sampling of the forcing function. Additionally, the inclusion of particular solutions related more intuitively to the physical situation of a problem can also be examined. Finally, the procedure can be extended for the solution of problems involving anisotropic materials.

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